

# HOW TO KICK A SOLITON ?

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A simple method how to study response of solitons in dissipative systems to external impulsive perturbations is developed. Thanks to nontrivial choice of small parameter, the perturbative scheme captures genuine nonlinear phenomena. The method is developed and tested by numerical simulations for kinks in 1+1 dimensions and for skyrmions in 2+1 dimensions. Extension to models including second order time derivatives is discussed.

Let us consider a dissipative version of the  $\phi^4$  theory in one spatial dimension, which is defined, in appropriate dimensionless units, by the nonlinear field equation

$$\partial_t \phi = \partial_x^2 \phi + 2[1 - \phi^2]\phi + \eta(t, x) . \quad (1)$$

In the absence of noise,  $\eta(t, x) = 0$ , Eq.(1) admits static kink solutions

$$\phi(t, x) = F(x) \equiv \tanh(x) . \quad (2)$$

Antikinks are given by  $F(-x)$ .

## Spectrum of kink excitations.

Small perturbations around the kink take the form  $e^{-\gamma t} g(x)$ . Linearization of Eq.(1) with respect to  $g(x)$ , for  $\eta(t, x) = 0$ , gives

$$\gamma g(x) = -\frac{d^2}{dx^2} g(x) + [4 - \frac{6}{\cosh^2(x)}] g(x) , \quad (3)$$

which looks like a stationary Schrödinger equation for a potential well problem. The eigenvalues and eigenstates can be tabulated as [1]

$$\begin{aligned} \gamma, & \quad g(x) , \\ 0, & \quad F'(x) = \frac{1}{\cosh^2(x)} , \\ 3, & \quad B(x) \equiv \frac{\sinh(x)}{\cosh^2(x)} , \\ 4 + k^2, & \quad g_k(x) \equiv e^{ikx} [1 + \frac{3ik \tanh(x) - 3 \tanh^2(x)}{1 + k^2}] , \end{aligned} \quad (4)$$

where  $k$  is a real momentum. The zero mode ( $\gamma = 0$ ) is separated by a gap from the first excited state ( $\gamma = 3$ ).

## Naive perturbation theory (NPT).

Let us consider the response of the kink to a solitary perturbation  $\eta(t, x) = \delta(t)\psi(x)$ . Let the unperturbed state be  $\phi(t < 0, x) = F(x)$ ; integration of Eq.(1) over time in an infinitesimal neighborhood of  $t = 0$  results in the initial configuration  $\phi(0^+, x) = F(x) + \psi(x)$ . The function  $\psi(x)$  can be expanded in the basis (4) as  $\psi(x) = \tilde{\psi}_z F'(x) + \tilde{\psi}_B B(x) + \int_{-\infty}^{\infty} dk \tilde{\psi}(k) g_k(x)$ . If  $\psi(x)$  is sufficiently small, such that Eq.(1) can be linearized around  $\phi(t, x) = F(x)$  for  $t > 0$ , then the perturbed field is well approximated by

$$\phi(t > 0, x) \approx F(x) + \tilde{\psi}_z F'(x) + e^{-3t} \tilde{\psi}_B B(x) + \int_{-\infty}^{\infty} dk e^{-(4+k^2)t} \tilde{\psi}(k) g_k(x) . \quad (5)$$

After a sufficiently long time, as compared to the relaxation time of the first excited state equal to  $1/3$ , the field relaxes to

$$\begin{aligned} \phi(\infty, x) & \approx F(x) + \tilde{\psi}_z F'(x) \approx F[x + \tilde{\psi}_z] , \\ \tilde{\psi}_z & = \frac{\int dx F'(x) \psi(x)}{\int dy F'(y) F'(y)} . \end{aligned} \quad (6)$$

The kink is shifted to a new position, the shift is given by a projection of the initial perturbation  $\psi(x)$  onto the kink's translational zero mode. However plausible it may sound, this result is not very accurate even for small perturbations. The reason is that the spectrum (4) is relevant for perturbations around the initial kink at the origin but it becomes less relevant as the kink drifts to its final location at  $-\tilde{\psi}_z$ . The excited states in (4) are not orthogonal to the zero mode  $F'[x + \tilde{\psi}_z]$  of the final kink.

## Improved perturbation theory (IPT).

An appropriate choice of perturbation theory may improve the accuracy. Let us consider again the single kick  $\eta(t, x) = \delta(t)\psi(x)$  and let us rewrite the initial perturbed field in the following form

$$\phi(0^+, x) \equiv F(x) + \psi(x) = F(x - \xi^+) + \mu(0^+, x) . \quad (7)$$

The above equation is just a tautology, unless we impose a constraint on the function  $\mu(0^+, x)$ . We require the function  $\mu(0^+, x)$  to be orthogonal to the zero mode of the kink at  $\xi^+$ ,

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$$0 = \int_{-\infty}^{+\infty} dx F'(x - \xi^+) \mu(0^+, x) . \quad (8)$$

This orthogonality implies that  $\mu(0^+, x)$  can be expressed as a combination of the excited states in the complete set (4) shifted to  $\xi^+$ ,

$$\mu(0^+, x) = \tilde{\mu}_B B(x - \xi^+) + \int_{-\infty}^{+\infty} dk \tilde{\mu}(k) g_k(x - \xi^+) . \quad (9)$$

So far we have made no approximation. The perturbative approximation is defined by the assumption that  $\mu(0^+, x)$  is sufficiently small, so that Eq.(1) can be linearized in  $\mu(t, x)$  around  $F(x - \xi^+)$ . The linearized solution is given by

$$\mu(t > 0, x) = \tilde{\mu}_B e^{-3t} B(x - \xi^+) + \int_{-\infty}^{+\infty} dk \tilde{\mu}(k) e^{-(4+k^2)t} g_k(x - \xi^+) . \quad (10)$$

After a sufficiently long time  $\mu(\infty, x) = 0$  and the initial condition on the R.H.S. of Eq.(7) evolves into  $F(x - \xi^+)$ . Thus  $\xi^+$  is the final position of the kink after perturbation.

To find  $\xi^+$ , let us project Eq.(7) on the zero mode  $F'(x - \xi^+)$ . Using Eq.(8) and the fact that  $F(x)$  is orthogonal to  $F'(x)$ , we obtain

$$\int_{-\infty}^{+\infty} dx F'(x - \xi^+) [F(x) + \psi(x)] = 0 , \quad (11)$$

which can be further rewritten as

$$U(\xi^+) + \int_{-\infty}^{+\infty} dx \frac{\psi(x + \xi^+)}{\cosh^2(x)} = 0 , \quad (12)$$

where  $U$  is the smooth, antisymmetric and monotonic function

$$U(\xi^+) = 2 \left[ \frac{1}{\tanh(\xi)} - \frac{\xi}{\sinh^2(\xi)} \right] . \quad (13)$$

Already from the general form of (12) it is clear, that  $\xi^+$  is not linearly dependent on the magnitude of the perturbation  $\psi(x)$ . This response is nonlinear although we use a sort of perturbation theory.

### Numerical simulation.

Before we proceed with general discussion, let us consider some quantitative examples and compare IPT predictions with direct numerical simulations of Eq.(1). We take the perturbation in the form

$$\psi(x) = -\frac{A}{\cosh^2(x - y)} , \quad (14)$$

where  $A$  is the amplitude and  $y$  the localization of the perturbation. Eq.(12) takes the form

$$A = \frac{\sinh^3(\xi^+ - y) \left[ \frac{1}{\tanh(\xi^+)} - \frac{\xi^+}{\sinh^2(\xi^+)} \right]}{2[(\xi^+ - y) \cosh(\xi^+ - y) - \sinh(\xi^+ - y)]} . \quad (15)$$

Fig.1 illustrates the dependence of the response  $\xi^+$  on the amplitude  $A$  for a central perturbation with  $y = 0$ . IPT works well up to  $A \approx 1$ , or up to soliton shifts comparable with soliton size. Fig.2 shows the data for  $A = 0.2$  and  $y$  varying in the range  $-2 < y < 2$ . If the perturbation hits to the right of the kink,  $y > 0$ , and is such that the kink is driven to the right,  $A > 0$ , the response is amplified because the perturbed kink is flowing towards the source of the perturbation. This positive feedback, which is missing in NPT, explains the asymmetry in Fig.2.

For some  $y$  Eq.(15) does not have a unique solution  $\xi^+ = \xi^+(A)$  in the whole range of  $A$ . This usually happens for  $|A| > 1$ , when the perturbation theory is not reliable a priori. As an example, Fig.3 illustrates the response for  $y = 2$  and  $A > 0$ . The theoretical function is not unique around  $A \approx 1.3$ . Numerical results coarse grain over this singularity. The perturbation (14) with  $y = 2$  creates a virtual antikink-kink pair just to the right of the original kink (2). The result is a kink-antikink-kink configuration. For a weak perturbation ( $A < 1.3$ ) the antikink annihilates with the right kink, leaving the original left kink shifted a bit to the right. For  $A > 1.3$  the antikink annihilates the original left kink, the net result is a huge shift to the right ( $\xi^+ > A$ ), much larger than that expected from NPT. This is again the positive feedback of Fig.2 but in a caricatured enhanced form. This is a genuine nonlinear response and it is amusing to find that IPT does capture its essential features.

### Response to subsequent perturbations.

$\xi^+$  is the position the kink relaxes to after a sufficiently long time. Before the final equilibrium is reached, the function  $\mu(t, x)$ , see Eq. (10), carries information about the initial perturbation. This information is lost after roughly 1/3 units of time, which is the relaxation time of the breather  $B(x)$  (4). As an example, let us consider the kink perturbed by the noise

$$\eta(t, x) = \delta(t) \psi(x) + \delta(t - T) \psi(x) , \quad (16)$$

where  $\psi(x)$  is given by (14) with, say,  $y = 0$  and  $A > 0$ . At the time  $t = T$  the field is hit for the second time,

$$\phi(T^+, x) = F(x - \xi^+) + \mu(T, x) + \psi(x) = F(x - \xi^{++}) + \bar{\mu}(T^+, x) , \quad (17)$$

while it still remembers the first perturbation  $\mu(T, x)$ . Repeating steps leading to (12), one obtains

$$U(\xi^{++} - \xi^+) + \int_{-\infty}^{+\infty} dx \frac{\psi(x + \xi^{++}) + \mu(T, x + \xi^{++})}{\cosh^2(x)} = 0 \quad (18)$$

There are two limiting cases. If  $T = 0$  the two kicks add up to a solitary perturbation  $2\psi(x)$ . If the function  $\xi^+(A, y)$  is a solution of Eq.(15), then the final shift of the kink can be expressed as  $\xi^{++}(T = 0) = \xi^+(2A, 0)$ . In the limit  $T = \infty$ , the kink has enough time to settle down at  $\xi^+(A, 0)$  after the first perturbation before it is shifted by the second kick to  $\xi^{++}(T = \infty) = \xi^+(A, 0) + \xi^+[A, -\xi^+(A, 0)]$ . The two shifts are different,  $\xi^{++}(T = \infty) > \xi(T = 0)$ . Fig.4 gives numerical results for various values of  $T$ . The limit  $\xi^{++}(T = \infty)$  is essentially achieved for  $T$  greater than the relaxation time of the breather mode (4),  $T > 1/3$ . For shorter intervals  $T$  the memory function  $\mu(T, x)$  in Eq.(18) cannot be neglected.

The second kick probes the position of the kink at the time  $T$  after the first perturbation. Roughly speaking, the kink remains unshifted for  $T < 1/3$ , it jumps to  $\xi^+(A, 0)$  for  $T > 1/3$ . The  $\delta(t)$ -like time shape of our impulse may be good approximation for impulses lasting less than  $1/3$ .

#### Higher order time derivatives.

IPT can be generalized to models with higher order time derivatives. For example, a modification

$$\partial_t \phi \rightarrow \partial_t \phi + \alpha \partial_t^2 \phi \quad (19)$$

on the LHS of Eq.(1) changes the initial conditions after the perturbation  $\eta(t, x) = \delta(t)\psi(x)$  to:  $\phi(0^+, x) = F(x)$ ,  $\partial_t \phi(0^+, x) = \psi(x)$ . The perturbed field can be written as  $\phi(t > 0, x) = F(x - \xi^+) + (A_1 + A_2 e^{-t/\alpha})F'(x - \xi^+) + R(t, x)$ .  $R(t, x)$  can be expanded in the excited states which decay with time. For  $\xi^+$  to be the final kink position,  $\phi(\infty, x) = F(x - \xi^+)$ , we must demand  $A_1 = 0$ . This constraint plus the initial conditions result in the following equation for  $\xi^+$

$$U(\xi^+) + \alpha \int_{-\infty}^{+\infty} dx \frac{\psi(x + \xi^+)}{\cosh^2(x)} = 0 \quad (20)$$

which is very similar to Eq.(12). It is important to realize that the  $\alpha \rightarrow 0$  limit of Eq.(20) does not give Eq.(12), which holds for  $\alpha = 0$ . The more general model is discontinuous at  $\alpha = 0$ .

We have performed several numerical simulations to test Eq.(20). The accuracy is the same as for  $\alpha = 0$ .

#### IPT and skyrmions in 2+1 dimensions.

Once IPT has passed the tests for kinks, it would be interesting to check how it works in higher dimensional

models. Let us consider the diffusion equation in the  $\vec{x} = (x_1, x_2)$  plane

$$\partial_t \hat{M} = P_{\hat{M}} \left[ -\frac{\delta W}{\delta \hat{M}} + \hat{\eta}(t, \vec{x}) \right] \quad (21)$$

where  $\hat{M}(t, \vec{x})$  is a unit ( $\hat{M}\hat{M} = 1$ ) magnetization vector,  $P_{\hat{M}}$  is a projector on the subspace orthogonal to  $\hat{M}$ ,  $W = \int d^2x w$  is the static energy functional with the energy density

$$w = \frac{1}{2} \partial_k \hat{M} \partial_k \hat{M} + \frac{1}{2} a_1 (M_1^2 + M_2^2) + \frac{1}{4} a_2 [(\partial_k \hat{M} \partial_k \hat{M})^2 - (\partial_k \hat{M} \partial_l \hat{M})^2] \quad (22)$$

being a sum of the exchange, easy axis anisotropy and Skyrme energy densities respectively. The energy is minimized by the skyrmion configuration [2]

$$\hat{F}(x_1, x_2) = \{\sin[q(r)] \cos \theta, \sin[q(r)] \sin \theta, \cos[q(r)]\} \quad (23)$$

where  $(r, \theta)$  are polar coordinates in the  $(x_1, x_2)$  plane and  $q(r)$  is a profile such that  $q(0) = \pi$  and  $q(\infty) = 0$ .  $q(r)$  has been determined by a shooting method for  $a_1 = 1$  and  $a_2 = 0.1$ , the result is shown in Fig.5.

The initial field after the perturbation  $\hat{\eta}(t, \vec{x}) = \delta(t)\hat{\psi}(\vec{x})$  is a bit complicated because of the projector on the RHS of Eq.(21). At any point  $\vec{x}$  the integration of Eq.(21) around  $t = 0$  gives the initial perturbed magnetization

$$c_- \equiv \frac{\hat{\psi}}{|\hat{\psi}|} \hat{F} \quad , \quad c_+ \equiv \frac{c_- + \tanh(|\hat{\psi}|)}{1 + c_- \tanh(|\hat{\psi}|)} \quad ,$$

$$\hat{M}(t = 0^+) = c_+ \frac{\hat{\psi}}{|\hat{\psi}|} + \sqrt{\frac{1 - c_+^2}{1 - c_-^2}} P_{\frac{\hat{\psi}}{|\hat{\psi}|}} [\hat{F}] \quad (24)$$

Similarly as for kinks, compare Eqs.(7,11), the final position  $\vec{\xi}^+$  of the skyrmion can be found from the condition that the initial perturbed field is orthogonal to the  $k = 1, 2$  translational zero modes of the final skyrmion

$$0 = \int d^2x \hat{M}(0^+, \vec{x}) \frac{\partial}{\partial x_k} \hat{F}(\vec{x} - \vec{\xi}^+) \quad (25)$$

We performed numerical simulations of Eq.(21) for the kick  $\hat{\psi}(\vec{x}) = -0.3 \frac{\partial}{\partial x_1} \hat{F}(\vec{x} - \vec{y})$  with  $\vec{y} = (y_1, 0)$  restricted to the  $x_1$ -axis. As a result of the perturbation the skyrmion was shifted to a new position at  $\vec{\xi}^+ = (\xi_1^+, 0)$ . The function  $\xi_1^+(y_1)$  is presented in Fig.6. The maximum of the function is shifted to the right and the minima are shifted to the left as a result of similar positive feedback as in Fig.2.

IPT works well for skyrmions in 2+1 dimensions. The model (21) can be generalized to, say, Landau-Lifshitz equation with dissipation. Modification of the LHS of Eq.(21),  $\partial_t \hat{M} \rightarrow \partial_t \hat{M} + \lambda \hat{M} \times \partial_t \hat{M}$ , changes  $\hat{M}(0^+, \vec{x})$  but the condition (25) remains the same.

### Remarks.

The theory outlined above can be upgraded to a fully fledged theory of soliton diffusion driven by a random noise. Another possible line of development is a generalization to the case of dissipative multisolitons [3].

### ACKNOWLEDGMENTS

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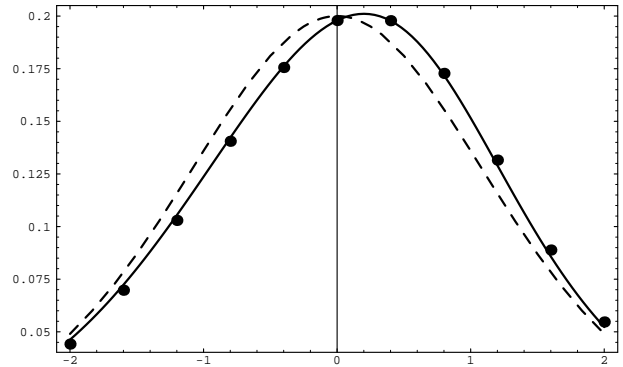


FIGURE 2.  $\xi^+$  for the kink as a function of  $y$  for  $A = 0.2$ .

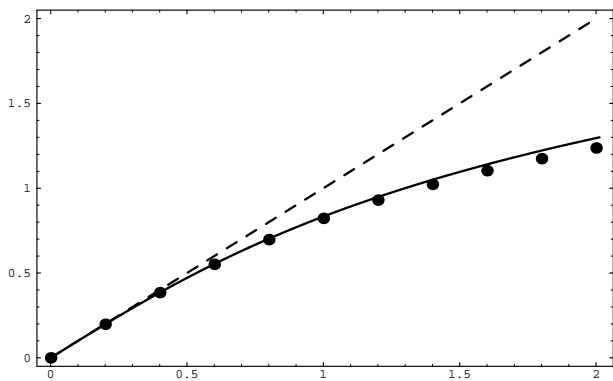


FIGURE 1.  $\xi^+$  for the kink as a function of  $A$  for  $y = 0$ . Dashed line - NPT, solid line - IPT and dots - numerical simulation.

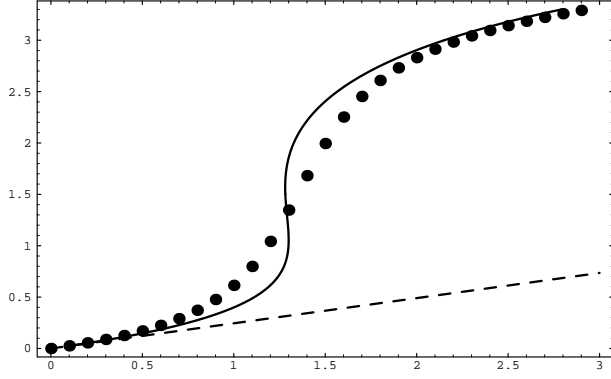


FIGURE 3.  $\xi^+$  for the kink as a function of  $A$  for  $y = 2$ .

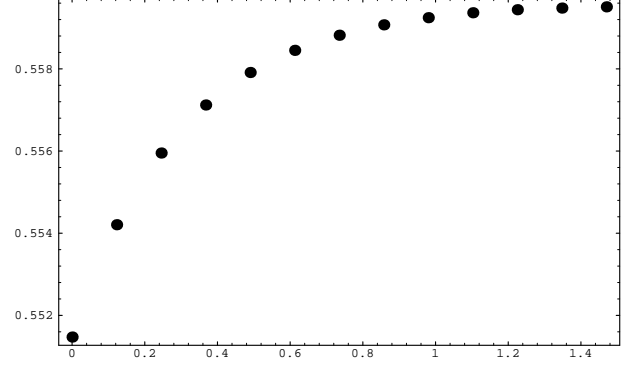


FIGURE 4. Numerical simulation results for  $\xi^{++}$  after two subsequent perturbations separated by  $T$  as a function of  $T$ . Amplitude  $A=0.3$ .

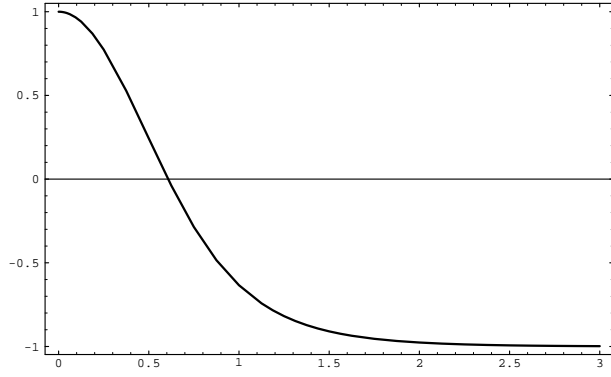


FIGURE 5. The magnetization  $M_3 = \cos[q(r)]$  as a function of  $r$ .

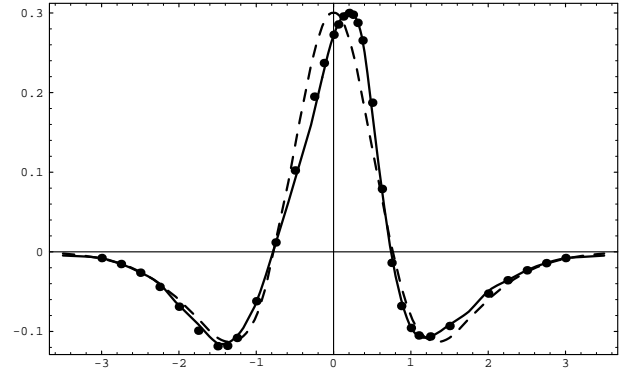


FIGURE 6.  $\xi_1^+$  for the skyrmion as a function of  $y_1$  for  $A = 0.3$ . Dashed line - NPT, solid line - IPT and dots - numerical simulation.